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## DISCRETE MODELS OF THE SELF-DUAL AND ANTI-SELF-DUAL EQUATIONS

In the case of a gauge-invariant discrete model of Yang-Mills theory difference self-dual and anti-self-dual equations are constructed.

### 1. Introduction.

In 4-dimensional non-abelian gauge theory the self-dual and anti-self-dual connections are the most important extrema of the Yang-Mills action. Consider a trivial bundle $P = \mathbb{R}^4 \times G$, where $G$ is some Lie group. We define a connection as some $\mathfrak{g}$-valued 1-form $A$, where $\mathfrak{g}$ is the Lie algebra of the group $G$ [5]. Then the connection 1-form $A$ can be written as follows

$$A = \sum_{a,\mu} A_\mu^a(x)\lambda_a dx^\mu, \tag{1}$$

where $\lambda_a$ is the basis of the Lie algebra $\mathfrak{g}$. The curvature 2-form $F$ of the connection $A$ is given by

$$F = dA + A \wedge A. \tag{2}$$

We specialize straightaway to the choice $G = SU(2)$, then $\mathfrak{g} = su(2)$. We define the covariant exterior differentiation operator $d_A$ by

$$d_A\Omega = d\Omega + A \wedge \Omega + (-1)^{r+1}\Omega \wedge A, \tag{3}$$

where $\Omega$ is an arbitrary $su(2)$-valued $r$-form. Compare (2) and (3) we obtain the Bianchi identity

$$d_AF = 0. \tag{4}$$

The Yang-Mills action $S$ can be conveniently expressed (see [5, p. 256]) in terms of the 2-forms $F$ and $*F$ as

$$S = -\int_{\mathbb{R}^4} tr(F \wedge *F),$$

where $*$ is the adjoint operator (Hodge star operator). The Euler-Lagrange equations for the extrema of $S$ are

$$d_A * F = 0. \tag{5}$$

Equations (4), (5) are called the Yang-Mills equations [4]. These equations are non-linear coupled partial differential equations containing quadratic and cubic terms in $A$.

In more traditional form the Yang-Mills equations are expressed in terms of components of the connection $A$ and the curvature $F$ (see [2,3]). Let

$$A_\mu = \sum_\alpha A_\mu^\alpha(x)\lambda_\alpha$$

be the component of the connection 1-form (1). Then the components of the curvature form are given by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} + \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu],$$

where $[\cdot\ ,\cdot]$ be the commutator of the algebra Lie $su(2)$. In local coordinates the covariant derivative $\nabla_j$ can be written

$$\nabla_j F_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial x^j} + [A_j, F_{\mu\nu}].$$

Then we can write Equations (4), (5) as

$$\nabla_j F_{\mu\nu} + \nabla_\mu F_{\nu j} + \nabla_\nu F_{j\mu} = 0, \tag{6}$$

$$\sum_{\mu=1} \frac{\partial F_{\mu\nu}}{\partial x^\mu} + [A_\mu, F_{\mu\nu}] = 0. \tag{7}$$

Note that Equations (7) are obtained in the case of Euclidean space $\mathbb{R}^4$.

The self-dual and anti-self-dual connections are solutions of the following nonlinear first order differential equations

$$F = *F, \qquad F = - * F. \tag{8}$$

Equations (8) are called self-dual and anti-self-dual respectively. It is obviously that if one can find $A$ such that $F = \pm * F$, then the Yang-Mills equations (5) are automatically satisfied.

## 2. The discrete model in $\mathbb{R}^4$.

In [6] the gauge invariant discrete model of the Yang-Mills equations is constructed in the case of the $n$-dimensional Euclidean space $\mathbb{R}^n$. Following [6], we consider a combinatorial model of $\mathbb{R}^4$ as a certain 4-dimensional complex $C(4)$. Let $K(4)$ be a dual complex of $C(4)$. The complex $K(4)$ is a 4-dimensional complex of cochains with $su(2)$-valued coefficients. We define the discrete analogs of the connection 1-form $A$ and the curvature 2-form $F$ as follows cochains

$$A = \sum_k \sum_{i=1}^4 A_k^i e_i^k, \qquad F = \sum_k \sum_{i<j} \sum_{j=2}^4 F_k^{ij} \varepsilon_{ij}^k, \tag{9}$$

where $A_k^i$, $F_k^{ij} \in su(2)$, $e_i^k$, $\varepsilon_{ij}^k$ are 1-, 2-dimensional basis elements of $K(4)$ and $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$. We use the geometrical construction proposed by A. A. Dezin in [1] to define discrete analogs of the differential, the exterior multiplication and the Hodge star operator.

Let us introduce for convenient the shifts operator $\tau_i$ and $\sigma_i$ as

$$\tau_i k = (k_1, ...\tau k_i, ...k_4), \qquad \sigma_i k = (k_1, ...\sigma k_i, ...k_4),$$

where $\tau k_i = k_i + 1$ and $\sigma k_i = k_i - 1$, $k_i \in \mathbb{Z}$. Similarly, we denote by $\tau_{ij}$ ($\sigma_{ij}$) the operator shifting to the right (to the left) two differ components of $k = (k_1, k_2, k_3, k_4)$. For example, $\tau_{12}k = (\tau k_1, \tau k_2, k_3, k_4)$, $\sigma_{14}k = (\sigma k_1, k_2, k_3, \sigma k_4)$.

If we use (2) and take the definitions of $d$ and $\wedge$ in discrete case [1,6], then we obtain

$$F_k^{ij} = \Delta_{k_i} A_k^j - \Delta_{k_j} A_k^i + A_k^i A_{\tau_i k}^j - A_k^j A_{\tau_j k}^i, \tag{10}$$

where $\Delta_{k_i} A_k^j = A_{\tau_i k}^j - A_k^j$, $i, j = 1, 2, 3, 4$. The metric adjoint operation $*$ acts on the 2-dimensional basis elements of $K(4)$ as follows

$$*\varepsilon_{12}^k = \varepsilon_{34}^{\tau_{12}k}, \qquad *\varepsilon_{13}^k = -\varepsilon_{24}^{\tau_{13}k}, \qquad *\varepsilon_{14}^k = \varepsilon_{23}^{\tau_{14}k},$$
$$*\varepsilon_{23}^k = \varepsilon_{14}^{\tau_{23}k}, \qquad *\varepsilon_{24}^k = -\varepsilon_{13}^{\tau_{24}k}, \qquad *\varepsilon_{34}^k = \varepsilon_{12}^{\tau_{34}k}.$$

Then we obtain

$$*F=\sum_k\big(F^{34}_{\sigma_{34}k}\varepsilon^k_{12}-F^{24}_{\sigma_{24}k}\varepsilon^k_{13}+F^{23}_{\sigma_{23}k}\varepsilon^k_{14}+ +F^{14}_{\sigma_{14}k}\varepsilon^k_{23}-F^{13}_{\sigma_{13}k}\varepsilon^k_{24}+F^{12}_{\sigma_{12}k}\varepsilon^k_{34}\big). \tag{11}$$

Comparing the latter and (9) the discrete analog of the self-dual equation (the first equation of (8)) we can written as follows

$$\begin{array}{lll} F^{12}_k=F^{34}_{\sigma_{34}k}, & F^{13}_k=-F^{24}_{\sigma_{24}k}, & F^{14}_k=F^{23}_{\sigma_{23}k}, \\ F^{23}_k=F^{14}_{\sigma_{14}k}, & F^{24}_k=-F^{13}_{\sigma_{13}k}, & F^{34}_k=F^{12}_{\sigma_{12}k} \end{array} \tag{12}$$

for all $k=(k_1,k_2,k_3,k_4),\ k_i\in\mathbb{Z}$. Using (10) Equations (12) can be rewritten in the following difference form:

$$\Delta_{k_1}A^2_k-\Delta_{k_2}A^1_k+A^1_k\cdot A^2_{\tau_1k}-A^2_k\cdot A^1_{\tau_2k}= =\Delta_{k_3}A^4_{\sigma_{34}k}-\Delta_{k_4}A^3_{\sigma_{34}k}+A^3_{\sigma_{34}k}\cdot A^4_{\sigma_4k}-A^4_{\sigma_{34}k}\cdot A^3_{\sigma_3k},$$

$$\Delta_{k_1}A^3_k-\Delta_{k_3}A^1_k+A^1_k\cdot A^3_{\tau_1k}-A^3_k\cdot A^1_{\tau_3k}= =-\Delta_{k_2}A^4_{\sigma_{24}k}+\Delta_{k_4}A^2_{\sigma_{24}k}-A^2_{\sigma_{24}k}\cdot A^4_{\sigma_4k}+A^4_{\sigma_{24}k}\cdot A^2_{\sigma_2k},$$

$$\Delta_{k_1}A^4_k-\Delta_{k_4}A^1_k+A^1_k\cdot A^4_{\tau_1k}-A^4_k\cdot A^1_{\tau_4k}= =\Delta_{k_2}A^3_{\sigma_{23}k}-\Delta_{k_3}A^2_{\sigma_{23}k}+A^2_{\sigma_{23}k}\cdot A^3_{\sigma_3k}-A^3_{\sigma_{23}k}\cdot A^2_{\sigma_2k},$$

$$\Delta_{k_2}A^3_k-\Delta_{k_3}A^2_k+A^2_k\cdot A^3_{\tau_2k}-A^3_k\cdot A^2_{\tau_3k}= =\Delta_{k_1}A^4_{\sigma_{14}k}-\Delta_{k_4}A^1_{\sigma_{14}k}+A^1_{\sigma_{14}k}\cdot A^4_{\sigma_4k}-A^4_{\sigma_{14}k}\cdot A^1_{\sigma_1k},$$

$$\Delta_{k_2}A^4_k-\Delta_{k_4}A^2_k+A^2_k\cdot A^4_{\tau_2k}-A^4_k\cdot A^2_{\tau_4k}= =-\Delta_{k_1}A^3_{\sigma_{13}k}+\Delta_{k_3}A^1_{\sigma_{13}k}-A^1_{\sigma_{13}k}\cdot A^3_{\sigma_3k}+A^3_{\sigma_{13}k}\cdot A^1_{\sigma_1k},$$

$$\Delta_{k_3}A^4_k-\Delta_{k_4}A^3_k+A^3_k\cdot A^4_{\tau_3k}-A^4_k\cdot A^3_{\tau_4k}= =\Delta_{k_1}A^2_{\sigma_{12}k}-\Delta_{k_2}A^1_{\sigma_{12}k}+A^1_{\sigma_{12}k}\cdot A^2_{\sigma_2k}-A^2_{\sigma_{12}k}\cdot A^1_{\sigma_1k}.$$

In the same way we obtain the difference anti-self-dual equation. From Equations (12) we obtain at once

$$F^{jr}_k=F^{jr}_{\sigma k} \tag{13}$$

for all $j<r,\ r=2,3,4$, where $\sigma k=(\sigma k_1,\sigma k_2,\sigma k_3,\sigma k_4)$.

Note that Equations (13) also are satisfied in the case of the difference anti-self-dual equations.

PROPOSITION 1. *Let $F$ be a solution of the discrete self-dual or anti-self dual equations. Then we have*

$$**F=F. \tag{14}$$

*Proof.* From (11) we have

$$**F=\sum_k\big(F^{34}_{\sigma_{34}k}*\varepsilon^k_{12}-F^{24}_{\sigma_{24}k}*\varepsilon^k_{13}+F^{23}_{\sigma_{23}k}*\varepsilon^k_{14}+ +F^{14}_{\sigma_{14}k}*\varepsilon^k_{23}-F^{13}_{\sigma_{13}k}*\varepsilon^k_{24}+F^{12}_{\sigma_{12}k}*\varepsilon^k_{34}\big)=$$

$$= \sum_k \left(F^{34}_{\sigma_{34}k}\varepsilon^{\tau_{12}k}_{34} + F^{24}_{\sigma_{24}k}\varepsilon^{\tau_{13}k}_{24} + F^{23}_{\sigma_{23}k}\varepsilon^{\tau_{14}k}_{23} + \right.$$
$$\left. + F^{14}_{\sigma_{14}k}\varepsilon^{\tau_{23}k}_{14} + F^{13}_{\sigma_{13}k}\varepsilon^{\tau_{24}k}_{13} + F^{12}_{\sigma_{12}k}\varepsilon^{\tau_{34}k}_{12}\right) =$$
$$= \sum_k \sum_{i<j} \sum_{j=2}^{4} F^{ij}_{\sigma k}\varepsilon^{k}_{ij}.$$

Comparing the latter and (13) we obtain (14).

□

It should be noted that in the case of continual Yang-Mills theory for $\mathbb{R}^4$ with the usual Euclidean metric Equation (14) is satisfied automatically for an arbitrary 2-form. But in the formalism we use the operation $(*)^2$ is equivalent to a shift.

The difference analog of Equations (13) is given by

$$\Delta_{k_j} A^r_k - \Delta_{k_r} A^j_k + A^j_k \cdot A^r_{\tau_j k} - A^r_k \cdot A^j_{\tau_r k} =$$
$$= \Delta_{k_j} A^r_{\sigma k} - \Delta_{k_r} A^j_{\sigma k} + A^j_{\sigma k} \cdot A^r_{\sigma\tau_j k} - A^r_{\sigma k} \cdot A^j_{\sigma\tau_r k},$$

where $\sigma\tau_j k = (\sigma k_1 ... k_j ... \sigma k_4)$.

**3. The discrete model in Minkowski space.**

Let a base space of the bundle $P$ be Minkowski space, i. e. $\mathbb{R}^4$ with the metric $g_{\mu\nu} = diag(- + ++)$. In Minkowski space we write Equations (8) as

$$*F = \mp iF, \tag{15}$$

where $i^2 = -1$. Recall that $F$ is $\mathfrak{g}$-valued, so therefore is $*F$. Then we must have $i\mathfrak{g} = \mathfrak{g}$ in obvious notation. However, this latter condition is not satisfied for the Lie algebras of any compact Lie groups $G$. To study Equations (15) we must choose non-compact $G$ such as $SL(n, \mathbb{C})$ or $GL(n, \mathbb{C})$ say. This is a serious restriction since in physics the gauge groups chosen are usually compact [5]. Let the gauge group be $G = SL(2, \mathbb{C})$.

We suppose that a combinatorial model of Minkowski space has the same structure as $C(4)$. A gauge-invariant discrete model of the Yang-Mills equations in Minkowski space is given in [7]. Now the dual complex $K(4)$ is a complex of $sl(2, \mathbb{C})$-valued cochains (forms). Because discrete analogs of the differential and the exterior multiplication are not depended on a metric then they have the same form as in the case of Euclidean space. For more details on this point see [7]. However, to define a discrete analog of the $*$ operation we must take into accounts the Lorentz metric structure on $K(4)$. We denote by $\bar{x}_\kappa,\ \bar{e}_\kappa,\ \kappa \in \mathbb{Z}$ the basis elements of the 1-dimensional complex $K$ which are corresponded to the time coordinate of Minkowski space. It is convenient to write the basis elements of $K(4) = K \otimes K \otimes K \otimes K$ in the form $\bar{\mu}^\kappa \otimes s^k$, where $\bar{\mu}^\kappa$ is either $\bar{x}^\kappa$ or $\bar{e}^\kappa$ and $s^k$ is a basis element of $K(3) = K \otimes K \otimes K$, $k = (k_1, k_2, k_3),\ \ \kappa, k_j \in \mathbb{Z}$. Then we define the $*$ operation on $K(4)$ as follows

$$\bar{\mu}^\kappa \otimes s^k \cup *(\bar{\mu}^\kappa \otimes s^k) = Q(\mu)\bar{e}^\kappa \otimes e^{k_1} \otimes e^{k_2} \otimes e^{k_3}, \tag{16}$$

where $Q(\mu)$ is equal to $+1$ if $\bar{\mu}^\kappa = \bar{x}^\kappa$ and to $-1$ if $\bar{\mu}^\kappa = \bar{e}^\kappa$. To arbitrary forms the $*$ operation is extended linearly. Using (16) we obtain

$$*F = \sum_k \left(F^{34}_{\sigma_{34}k}\varepsilon^{k}_{12} - F^{24}_{\sigma_{24}k}\varepsilon^{k}_{13} + F^{23}_{\sigma_{23}k}\varepsilon^{k}_{14} - \right.$$
$$\left. - F^{14}_{\sigma_{14}k}\varepsilon^{k}_{23} + F^{13}_{\sigma_{13}k}\varepsilon^{k}_{24} - F^{12}_{\sigma_{12}k}\varepsilon^{k}_{34}\right), \tag{17}$$

where $F_k^{ij} \in sl(2,\mathbb{C})$. Combining (17) with (9) the discrete self-dual equation $*F = iF$ can be written as follows

$$\begin{array}{lll} F^{34}_{\sigma_{34}k} = iF^{12}_k, & -F^{24}_{\sigma_{24}k} = iF^{13}_k, & F^{23}_{\sigma_{23}k} = iF^{14}_k, \\ -F^{14}_{\sigma_{14}k} = iF^{23}_k, & F^{13}_{\sigma_{13}k} = iF^{24}_k, & -F^{12}_{\sigma_{12}k} = iF^{34}_k \end{array} \tag{18}$$

for all $k = (k_1, k_2, k_3, k_4),\ k_r \in \mathbb{Z},\ r = 1,2,3,4$. From the latter we obtain

$$F^{34}_{\sigma k} = iF^{12}_{\sigma_{12}k} = -i^2 F^{34}_k = F^{34}_k, \qquad F^{24}_{\sigma k} = -iF^{13}_{\sigma_{13}k} = -i^2 F^{24}_k = F^{24}_k$$

and similarly for any other components $F^{jr}_k,\ j < r$. So we have Relations (13). Thus a solution of the discrete self-dual equations (18) satisfies Equations (13) as in the Euclidean case.

We can also rewrite (18) in the difference form

$$\begin{aligned} &\Delta_{k_3} A^4_{\sigma_{34}k} - \Delta_{k_4} A^3_{\sigma_{34}k} + A^3_{\sigma_{34}k} \cdot A^4_{\sigma_4 k} - A^4_{\sigma_{34}k} \cdot A^3_{\sigma_3 k} = \\ &\quad = i(\Delta_{k_1} A^2_k - \Delta_{k_2} A^1_k + A^1_k \cdot A^2_{\tau_1 k} - A^2_k \cdot A^1_{\tau_2 k}), \end{aligned}$$

$$\begin{aligned} &-\Delta_{k_2} A^4_{\sigma_{24}k} + \Delta_{k_4} A^2_{\sigma_{24}k} - A^2_{\sigma_{24}k} \cdot A^4_{\sigma_4 k} + A^4_{\sigma_{24}k} \cdot A^2_{\sigma_2 k} = \\ &\quad = i(\Delta_{k_1} A^3_k - \Delta_{k_3} A^1_k + A^1_k \cdot A^3_{\tau_1 k} - A^3_k \cdot A^1_{\tau_3 k}), \end{aligned}$$

$$\begin{aligned} &\Delta_{k_2} A^3_{\sigma_{23}k} - \Delta_{k_3} A^2_{\sigma_{23}k} + A^2_{\sigma_{23}k} \cdot A^3_{\sigma_3 k} - A^3_{\sigma_{23}k} \cdot A^2_{\sigma_2 k} = \\ &\quad = i(\Delta_{k_1} A^4_k - \Delta_{k_4} A^1_k + A^1_k \cdot A^4_{\tau_1 k} - A^4_k \cdot A^1_{\tau_4 k}), \end{aligned}$$

$$\begin{aligned} &-\Delta_{k_1} A^4_{\sigma_{14}k} + \Delta_{k_4} A^1_{\sigma_{14}k} - A^1_{\sigma_{14}k} \cdot A^4_{\sigma_4 k} + A^4_{\sigma_{14}k} \cdot A^1_{\sigma_1 k} = \\ &\quad = i(\Delta_{k_2} A^3_k - \Delta_{k_3} A^2_k + A^2_k \cdot A^3_{\tau_2 k} - A^3_k \cdot A^2_{\tau_3 k}), \end{aligned}$$

$$\begin{aligned} &\Delta_{k_1} A^3_{\sigma_{13}k} - \Delta_{k_3} A^1_{\sigma_{13}k} + A^1_{\sigma_{13}k} \cdot A^3_{\sigma_3 k} - A^3_{\sigma_{13}k} \cdot A^1_{\sigma_1 k} = \\ &\quad = i(\Delta_{k_2} A^4_k - \Delta_{k_4} A^2_k + A^2_k \cdot A^4_{\tau_2 k} - A^4_k \cdot A^2_{\tau_4 k}), \end{aligned}$$

$$\begin{aligned} &-\Delta_{k_1} A^2_{\sigma_{12}k} + \Delta_{k_2} A^1_{\sigma_{12}k} - A^1_{\sigma_{12}k} \cdot A^2_{\sigma_2 k} + A^2_{\sigma_{12}k} \cdot A^1_{\sigma_1 k} = \\ &\quad = i(\Delta_{k_3} A^4_k - \Delta_{k_4} A^3_k + A^3_k \cdot A^4_{\tau_3 k} - A^4_k \cdot A^3_{\tau_4 k}). \end{aligned}$$

In similar manner we obtain the difference anti-self-dual equations. Obviously an anti-self-dual solution satisfies Equations (13).

PROPOSITION 2. *Let for any $sl(2,\mathbb{C})$-valued 2-form $F$ Conditions (13) are satisfied. Then we have*

$$**F = -F.$$

*Proof.* If components of any discrete 2-form $F$ satisfy (13), then $F$ is a solution of the discrete self-dual or anti-self-dual equations. Hence

$$**F = *(\mp iF) = \mp i * F = (\mp i)^2 F = -F.$$

□

REMARK. *In the continual case the self-dual and anti-self-dual equations are written in the form (15) because we have* $**F = -F$ *for an arbitrary 2-form* $F$ *in Minkowski space. In the discrete model case it is easy to check that in* $K(4)$ *we have*

$$**F = -\sum_k \sum_{i<j} \sum_{j=2}^{4} F_{\sigma k}^{ij} \varepsilon_{ij}^k.$$

*Thus Equations (15) are satisfied only under Conditions (13).*

THEOREM. *If exist some* $N = (N_1, N_2, N_3, N_4)$, $N_r \in \mathbb{Z}$ *such that*

$$F_k^{ij} = 0 \quad for\ any \quad |k| \geq |N|, \tag{19}$$

*then Equations (15) (or (8)) have only the trivial solution* $F = 0$.

*Proof.* Since for any solution of Equations (15) (or (8)) we have Relations (13) then the assertion is obvious.

□

Let $g$ be a discrete 0-form

$$g = \sum_k g_k x^k,$$

where $x^k$ is the 0-dimensional basis element of $K(4)$ and $g_k \in SU(2)$ (or $g_k \in Sl(2, \mathbb{C})$). The boundary condition (19) in terms of the connection components can be represented as: there is some discrete 0-form $g$ such that

$$A_k^j = -(\Delta_{k_j} g_k) g_k^{-1} \quad \text{for any} \quad |k| \geq |N|.$$

It follows from Theorem 3 [6].


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